THE EIGENVALUE BEHAVIOR OF CERTAIN CONVOLUTION EQUATIONS

BY

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Introduction. In a series of papers [3], [4], [6], we studied the relationship between two closed subspaces of $L^2(-\infty, \infty)$: the subspace $\mathscr{D}_T$ of all $f \in L^2$ supported in $|t| < T/2$ and the subspace $\mathscr{B}_\Omega$ of all $f \in L^2$ whose Fourier transforms are supported in $|\omega| < \Omega/2$. We showed that several questions about $\mathscr{D}_T$ and $\mathscr{B}_\Omega$ could be answered in terms of the eigenvalues of the operator $B_\Omega D_T B_\Omega$, where B_Ω and D_T are the projections onto $\mathscr{B}_\Omega$ and $\mathscr{D}_T$ respectively; this operator may be written as a finite convolution. Apart from this application, interpretable as describing the way in which the energy of a function of L^2 can be distributed over time and over frequency, the behavior of these eigenvalues is interesting because it differs markedly from that established by H. Widom [7] for the class of finite convolutions with L^1 kernels whose Fourier transforms have an absolute maximum at the origin.

By a change of variable, the eigenvalues of $B_\Omega D_T B_\Omega$ may be seen to depend on the parameter $c = \Omega T/2\pi$, rather than on Ω and T separately; we may write their equation explicitly as

$$\lambda_n(c)\phi_n^{(c)}(t) = \frac{1}{\pi}\int_{-c/2}^{c/2} \frac{\sin \pi(t-x)}{t-x}\,\phi_n^{(c)}(x)\,dx, \qquad n = 0,1,2,\cdots,$$

and we suppose that $\lambda_0 \geqq \lambda_1 \geqq \cdots$. For any fixed c, the $\lambda_n(c)$, $n = 0, 1, \cdots$, form a positive sequence bounded away from 1 and approaching 0 at a rate in n greater than $(ce/n)^{2n}$ [D. Slepian, unpublished]. For any fixed n, the eigenvalue $\lambda_n(c)$ approaches 1 exponentially in c [2]. In [4] we proved, however, that $\lambda_{[c]+1}(c)$ is bounded away from 1 independently of c, and interpreted this to imply that the set of functions in $\mathscr{B}_\Omega$ whose energy is concentrated in $|t| < T/2$ has, in a well-defined sense, approximate dimension bounded by $[\Omega T/2\pi]$ (1). We also showed that $\lambda_{[c]-1}(c)$ is bounded away from 0 independently of c.

The analogous questions for the case where the intervals $|t| < T/2$ and $|\omega| < \Omega/2$ are replaced by more general sets T' and Ω' have not been studied. Indeed, most of the methods developed to deal with $\mathscr{B}_\Omega$ are not applicable to $\mathscr{B}_{\Omega'}$, and very little is known about it. Here we give another, simpler, proof that $\lambda_{[c]+1}(c)$ and $\lambda_{[c]-1}(c)$ are bounded away from 1 and 0 respectively,

Received by the editors December 2, 1963.

(1) $[x]$ denotes the largest integer less than or equal to x.

independently of c. Our method improves considerably on the bound of [4], but its most interesting feature is its extendability to the case that T' and Ω' are each finite unions of intervals.

ACKNOWLEDGEMENTS. I am greatly indebted to H. O. Pollak, D. Slepian, and L. J. Wallen for many interesting conversations.

PRELIMINARIES. We consider the Hilbert space $L^2(-\infty, \infty)$ with the usual definition of the scalar product

$$(f,g) = \int_{-\infty}^{\infty} f(t)\overline{g(t)}\, dt,$$

and denote by $F(\omega)$ the Fourier transform of $f(t) \in L^2$,

$$F(\omega) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt.$$

If S is an open subset of the real line, we single out two subspaces of L^2:

$$\mathscr{D}(S) = \{f \in L^2 | f(t) \equiv 0,\ t \notin S\},$$

$$\mathscr{B}(S) = \{f \in L^2 | F(\omega) \equiv 0,\ \omega \notin S\}.$$

If S is a bounded set, and $f \in \mathscr{B}(S)$, writing $f(t)$ as the inverse transform of $F(\omega)$ exhibits f as the restriction to the reals of an entire function $f(t+iu)$ of exponential type. If $\|f\| = 1$, Schwarz's inequality and Parseval's theorem applied to this representation show that $f(t+iu)$ is bounded in any given horizontal strip, by a constant depending only on the strip.

$\mathscr{D}(S)$ and $\mathscr{B}(S)$ are closed, since the Fourier transform preserves the norm. Let D_S and B_S denote the orthogonal projection operators of L^2 onto $\mathscr{D}(S)$ and $\mathscr{B}(S)$ respectively; their concrete representation is

$$D_S f(t) = \chi_S(t) f(t),$$

$$B_S f(t) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} \chi_S(\omega) F(\omega) e^{i\omega t} d\omega,$$

where $\chi_S(u)$ is the characteristic function of S, i.e.,

$$\chi_S(u) = \begin{cases} 1, & u \in S \\ 0, & u \notin S. \end{cases}$$

If S and S' are two open sets, the operator $B_S D_{S'} B_S$ is bounded by 1, self-adjoint, and positive. If S and S' have finite measure we may write $B_S D_{S'}$ explicitly as

$$B_S D_{S'}\phi = (2\pi)^{-1/2}\int_{-\infty}^{\infty} \chi_{S'}(u) h(t-u)\phi(u)\, du,$$

where the Fourier transform of h coincides with $\chi_S(\omega)$. Since, by Parseval's theorem,

$$\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dt |\chi_{S'}(u)h(t-u)|^2 = \int_{-\infty}^{\infty} |\chi_{S'}(u)|^2 du \int_{-\infty}^{\infty} |\chi_S(\omega)|^2 d\omega < \infty,$$

the operator $B_S D_{S'}$ is completely continuous [5, p. 159], and hence so is $B_S D_{S'} B_S$. Consequently [5, p. 233] the spectrum of $B_S D_{S'} B_S$ consists of isolated positive eigenvalues, bounded by 1 and accumulating at zero. In concrete form, the eigenvalue equation is a convolution:

$$\lambda_n \phi_n(t) = (2\pi)^{-1/2} \int_{x \in S'} \phi_n(x) h(t-x)\, dx.$$

RESULTS.

THEOREM 1. *Let P and Q be intervals of lengths W and T respectively, and let $c = WT/2\pi$. Let $\lambda_0, \lambda_1, \cdots$ be the eigenvalues of $B_P D_Q B_P$, arranged in nonincreasing order. Then* $\lambda_{[c]+1} \leqq .6$.

Proof. By a change of scale on the sets P and Q which does not alter c or the eigenvalues λ_i we may normalize the problem so that $W = 2\pi$, $T = c$, P coincides with $|\omega| < \pi$ and Q with $-1/2 < t < c - 1/2$. To simplify notation we henceforth drop the subscripts on B_P and D_Q.

The Weyl-Courant lemma [5, p. 238] asserts that

$$\lambda_{[c]+1} \leqq \sup_{(f,\psi_i)=0} \frac{(BDBf, f)}{\|f\|^2}, \tag{1}$$

where ψ_i, $i = 0, \cdots, [c]$, are any $[c]+1$ functions of L^2. Since B and D are orthogonal projections, they are self-adjoint and idempotent operators, so that

$$(BDBf, f) = (D^2Bf, Bf) = \|DBf\|^2.$$

Furthermore $\|f\|^2 \geqq \|Bf\|^2$, with equality equivalent to $f = Bf$, that is to $f \in \mathscr{B}(P)$. Consequently we may rewrite (1) as

$$\lambda_{[c]+1} \leqq \sup_{f \in \mathscr{B}(P);\, (f,\psi_i)=0} \frac{\|Df\|^2}{\|f\|^2}, \qquad i = 0, \cdots, [c]. \tag{2}$$

Let $h(t) \in L^2$ vanish for $|t| \geqq 1/2$ and let its Fourier transform $H(\omega)$ satisfy

$$|H(\omega)| \geqq 1, \qquad |\omega| \leqq \pi. \tag{3}$$

Now given $f \in \mathscr{B}(P)$, we consider the function

$$g(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(t)h(x-t)\, dt = (2\pi)^{-1/2} \int_{|t-x|<1/2} f(t)h(x-t)\, dt, \tag{4}$$

whose Fourier transform is $F(\omega)H(\omega)$. Since $f \in \mathscr{B}(P)$, the function $F(\omega)$

vanishes for $|\omega| \geqq \pi$, and $H(\omega)$ is bounded, so that $F(\omega)H(\omega) \in L^2(-\pi,\pi)$. Let a_k be the $(-k)$th coefficient of the Fourier series expansion of $F(\omega)H(\omega)$ in $|\omega| < \pi$. By definition

$$(5) \quad a_k = (2\pi)^{-1/2}\int_{-\pi}^{\pi} F(\omega)H(\omega)e^{ik\omega}d\omega = (2\pi)^{-1/2}\int_{-\infty}^{\infty} F(\omega)H(\omega)e^{ik\omega}d\omega = g(k),$$

whence using Parseval's theorem and (3)

$$(6) \quad \|f\|^2 = \int_{-\pi}^{\pi} |F(\omega)|^2 d\omega \leqq \int_{-\pi}^{\pi} |F(\omega)H(\omega)|^2 d\omega = \sum_{-\infty}^{\infty} |a_k|^2 = \sum_{-\infty}^{\infty} |g(k)|^2.$$

Next let us in (2) set

$$(7) \qquad \psi_i(t) = \overline{h(i-t)}, \qquad i = 0, 1, \cdots, [c].$$

By (4) the conditions $(f, \psi_i) = 0$ are equivalent to $g(k) = 0$ for $k = 0, \cdots, [c]$, so that from (6)

$$(8) \qquad \|f\|^2 \leqq \sum_{k<0;k>[c]} |g(k)|^2.$$

Schwarz's inequality applied to (4) yields

$$(9) \qquad |g(k)|^2 \leqq \frac{\|h\|^2}{2\pi} \int_{|t-k|<1/2} |f(t)|^2 dt,$$

and the intervals $|t-k| < 1/2$ with $k < 0$ and $k > [c]$ all lie outside Q. Hence, combining (8) and (9) gives

$$(10) \qquad 2\pi\|f\|^2 \leqq \|h\|^2 \int_{t\notin Q} |f(t)|^2 dt = \|h\|^2[\|f\|^2 - \|Df\|^2],$$

whence

$$(11) \qquad \|Df\|^2/\|f\|^2 \leqq 1 - 2\pi/\|h\|^2.$$

Letting $h(t) = \pi(\pi/2)^{1/2}$ in $|t| \leqq 1/2$ and 0 elsewhere, (2) and (11) prove

$$\lambda_{[c]+1} \leqq 1 - 4/\pi^2 < .6.$$

Theorem 1 is established.

H. O. Pollak has shown that, as in [4], the argument of Theorem 1 can serve as a basis for establishing a lower bound independent of c for $\lambda_{[c]-1}$.

THEOREM 2 (H. O. POLLAK). *Under the hypotheses of Theorem* 1, $\lambda_{[c]-1} \geqq .4$.

Proof. A simple consequence of the Weyl-Courant lemma is

$$(12) \qquad \lambda_{[c]-1} \geqq \inf_{f \in S_{[c]}} \frac{(BDBf, f)}{\|f\|^2},$$

where $S_{[c]}$ is any $[c]$-dimensional subspace of L^2. If we choose $S_{[c]}$ to be a subspace of $\mathcal{B}(P)$ we may, following Theorem 1, rewrite (12) as

$$\lambda_{[c]-1} \geqq \inf_{f\in \mathcal{B}(P);f\in S_{[c]}} \frac{\|Df\|^2}{\|f\|^2}. \tag{13}$$

Now with the function $h(t)$ of (3), we let $S_{[c]} \subset \mathcal{B}(P)$ be the subspace spanned by the $[c]$ (independent) members of $\mathcal{B}(P)$ whose Fourier transforms are $(2\pi)^{-1/2}e^{-ik\omega}/H(\omega)$ on $|\omega| \leqq \pi$, with $k = 0, \cdots, [c]-1$. Then by definition, if $f\in S_{[c]}$, its Fourier transform $F(\omega)$ satisfies

$$H(\omega)F(\omega) = \sum_{k=0}^{[c]-1} b_k(2\pi)^{-1/2}e^{-ik\omega}, \qquad |\omega| \leqq \pi, \tag{14}$$

so that, as in Theorem 1, we have $H(\omega)F(\omega)$ written as a Fourier series. Then, as in (5), (6), and (9)

$$\|f\|^2 \leqq \sum_0^{[c]-1} |b_k|^2, \tag{15}$$

$$|b_k|^2 = \left| (2\pi)^{-1/2}\int_{|t-k|<1/2} f(t)h(k-t)\,dt \right|^2 \leqq \frac{\|h\|^2}{2\pi}\int_{|t-k|\leqq 1/2} |f(t)|^2 dt, \tag{16}$$

but now the intervals $|t-k| \leqq 1/2$ for $k = 0, \cdots, [c]-1$ all lie inside Q. Hence, combining (15) and (16) gives

$$2\pi\|f\|^2 \leqq \|h\|^2 \int_{t\in Q} |f(t)|^2 dt = \|h\|^2\|Df\|^2, \tag{17}$$

which implies $\|Df\|^2/\|f\|^2 \geqq 2\pi/\|h\|^2$ for all $f\in S_{[c]}$. By (13), choosing the $h(t)$ of Theorem 1,

$$\lambda_{[c]-1} \geqq .4.$$

Theorem 2 is established.

B. F. Logan has proved [to appear] that, by proper choice of $h(t)$, the bounds of Theorems 1 and 2 can be improved to .5; together with Lemma 2 this implies that $\lim_{c\to\infty}\lambda_{[c]}(c) = 1/2$. We showed in [4] that the change in size of λ_k from near 1 to near 0 occurs in a strip around $k = [c]$ which grows no faster than $\log c$ but also does not remain bounded. Thus the description of the eigenvalues seems fairly complete.

Theorems 1 and 2 possess extensions to the case where the sets P and Q are finite unions of intervals.

LEMMA 1. *Let S consist of the union of m disjoint intervals and have total measure M. Then*

a. *the number $N(S)$ of integers k for which the interval $|k-t| < 1/2$ inter-*

sects S does not exceed $[M]+2m$;

b. *the number* $N'(S)$ *of integers* k *for which the interval* $|k-t|<1/2$ *is contained in* S *exceeds* $M-2m$.

Proof. If S is a single interval, let it coincide with $\alpha<t<M+\alpha$, let k_1 be the least integer with $k_1+1/2>\alpha$ and k_2 the largest integer with $k_2-1/2<M+\alpha$. Then $N(S)=k_2-k_1+1<M+2$. Now if S is the union of disjoint intervals S_i of measures m_i, with $i=1,\cdots,m$, then $N(S) \leqq \sum_i N(S_i) < \sum_i\{M_i+2\}=M+2m$, consequently $N(S) \leqq [M]+2m$.

Similarly, when S is a single interval, let k_1' be the largest integer with $k_1'-1/2<\alpha$ and k_2' the least integer with $k_2'+1/2>M+\alpha$. Then $N'(S)=k_2'-k_1'-1>M-2$, and if S is the union of m disjoint intervals $N'(S)=\sum_i N'(S_i)>\sum_i\{M_i-2\}=M-2m$. Lemma 1 is established.

Corollary 1. *Let one of the sets* P *and* Q *of measures* W *and* T *respectively, be a single interval, and the other be the union of* m *disjoint intervals. With* c *and* λ_n *defined as in Theorem* 1, $\lambda_{[c]+2m}<.6$ *and* $\lambda_{[c]-2m}>.4$.

Proof. The spectra of BDB and DBD are identical. For if $BDB\phi=\lambda B\phi$, we apply D to both sides and use the idempotency of D to obtain $DBD(DB\phi) = \lambda(DB\phi)$, and conversely. Consequently, in the proof of Theorems 1 and 2, the roles of t and ω are interchangeable. We may accordingly suppose that P consists of a single interval, which we normalize as before to coincide with $|\omega|<\pi$, whereupon Q becomes the union of m disjoint intervals of total measure c. Let $N(Q)$ and $N'(Q)$ be as in Lemma 1. We may now in the proof of Theorem 1 replace $[c]+1$ by $[c]+2m$ in (1) and argue without further change until (7), where we choose $\psi_i(t)=\overline{h(i-t)}$ for those i counted by $N(Q)$. By Lemma 1, the number of functions so obtained does not exceed $[c]+2m$, inequality (10) follows as before, and we may repeat the rest of the argument to show that $\lambda_{[c]+2m}<.6$. Similarly, in the proof of Theorem 2 we replace $[c]-1$ by $[c]-2m$ in (12) and $S_{[c]}$ by the subspace S of $\mathscr{B}(P)$ spanned by the functions whose Fourier transforms on $|\omega|\leqq\pi$ are $(2\pi)^{-1/2}e^{-ik\omega}/H(\omega)$ for those k counted by $N'(Q)$. By Lemma 1, the number of functions so obtained exceeds $c-2m$ and, being an integer, is no smaller than $[c]-2m+1$. Hence S has dimension at least $[c]-2m+1$, and we may repeat the rest of the argument to show that $\lambda_{[c]-2m}>.4$. Corollary 1 is established.

Theorem 3. *Let* P *and* Q *be unions of* p *and* q *disjoint intervals of total measure* W *and* T *respectively. Let* c *and* λ_n *be defined as in Theorem* 1. *Then* $\lambda_{[c]+2pq}\leqq J<1$, *where* J *is a constant depending only on* P (*suitably normalized*) *but not on* Q.

Proof. We may again suppose $W=2\pi$ and $T=c$, since this may always be achieved by a change of scale on the sets P and Q which does not alter c, p, q or the eigenvalues λ_n, and we drop the subscripts of B_P and D_Q. Let us denote by $\sigma_1, \cdots, \sigma_p$ the disjoint intervals comprising P, let $2\pi l_N$ be the length of σ_N, and $\chi_N(\omega)$ be its characteristic function. Let $h_N(t)$ be a function of L^2 which vanishes for $|t| \geqq 1/2l_N$ and whose Fourier transform $H_N(\omega)$ satisfies

$$(18) \qquad |H_N(\omega) - \chi_N(\omega)| < \epsilon, \qquad \omega \in P,$$

with $\epsilon = \{32p^2 + 16\sum_N[1/l_N]\}^{-1/2}$. Such an $h_N(t)$ exists, since $\chi_N(\omega)$ is continuous on P, and Fourier transforms of functions vanishing outside any fixed interval are uniformly dense in continuous functions on any compact set. Let $\gamma_N = \|h_N\|^2$.

As in Theorem 1, we will base our proof on the Weyl-Courant lemma, which asserts that

$$(19) \qquad \lambda_{[c]+2pq} \leqq \sup_{f \in \mathscr{B}(P);\,(f,\psi_i)=0} \frac{\|Df\|^2}{\|f\|^2},$$

where ψ_i, $i = 1, \cdots, [c] + 2pq$, are any $[c] + 2pq$ functions of L^2. Now given $f \in \mathscr{B}(P)$ with Fourier transform $F(\omega)$ we consider the function

$$(20)\ g_N(x) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} f(t)h_N(x-t)\,dt = (2\pi)^{-1/2}\int_{|t-x|<1/2l_N} f(t)h_N(x-t)\,dt,$$

whose Fourier transform is $F(\omega)H_N(\omega)$. By definition and a change of variable

$$\begin{aligned}(l_N)^{-1/2}g_N(k/l_N) &= (2\pi l_N)^{-1/2}\int_P F(\omega)H_N(\omega)\exp(i\omega k/l_N)\,d\omega \\ &= (2\pi l_N)^{-1/2}\int_P \sum_r \chi_N(\omega + r2\pi l_N)F(\omega)H_N(\omega)\exp(i\omega k/l_N)\,d\omega \\ (21) \qquad &= (2\pi l_N)^{-1/2}\int_{\sigma_N}\left\{\sum_r F(\omega - r2\pi l_N)H_N(\omega - r2\pi l_N)\right\}\exp(i\omega k/l_N)\,d\omega \\ &= \int_{\sigma_N}\{F(\omega) + R_N(\omega)\}(2\pi l_N)^{-1/2}\exp(i\omega k/l_N)\,d\omega,\end{aligned}$$

where

$$(22) \quad R_N(\omega) = \sum_r F(\omega - r2\pi l_N)\{H_N(\omega - r2\pi l_N) - \chi_N(\omega - r2\pi l_N)\}, \qquad \omega \in \sigma_N.$$

Since the functions $(2\pi l_N)^{-1/2}\exp(i\omega k/l_N)$ form a complete orthonormal set on σ_N, (21) implies

$$l_N^{-1} \sum_{k=-\infty}^{\infty} \left| g_N\left(\frac{k}{l_N}\right)\right|^2 = \int_{\sigma_N} | F(\omega) + R_N(\omega)|^2 d\omega. \tag{23}$$

To estimate the size of $R_N(\omega)$ on σ_N, we observe that the summands which do not vanish identically on σ_N correspond to those r for which the translate of σ_N by $r2\pi l_N$ intersects P. By Lemma 1, applied with a change of scale, the number of such terms does not exceed $[1/l_N] + 2p$. Hence Schwarz's inequality and (18) applied to (22) yield

$$\begin{aligned} \int_{\sigma_N} |R_N(\omega)|^2 d\omega \leqq \{ [1/l_N] + 2p \} & \\ \int_{\sigma_N} \sum_r | F(\omega - r2\pi l_N)\{ H_N(\omega - r2\pi l_N) &- \chi_N(\omega - r2\pi l_N) \}|^2 d\omega \\ = \{ [1/l_N] + 2p \} \int_P | F(\omega) \{ H_N(\omega) &- \chi_N(\omega) \}|^2 d\omega \\ \leqq \epsilon^2 \{ [1/l_N] + 2p \} \| F \|^2. & \end{aligned} \tag{24}$$

Also, by Schwarz's inequality applied to (20),

$$|g_N(k/l_N)|^2 \leqq \frac{\gamma_N}{2\pi} \int_{|t-k/l_N| < 1/2l_N} |f(t)|^2 dt. \tag{25}$$

Now, again by Lemma 1, the number of points k/l_N for which the interval $|t - k/l_N| < 1/2l_N$ intersects Q does not exceed $[cl_N] + 2q$. As in Theorem 1, we will require $f(t)$ to be orthogonal to the function $\overline{h_N(k/l_N - t)}$ for each such k/l_N; by (20) this is equivalent to the vanishing of $g_N(k/l_N)$. Thus by imposing no more than $[cl_N] + 2q$ orthogonality conditions on f, the sum on the left-hand side of (23) is extended over only those k for which the interval $|t - k/l_N| < 1/2l_N$ lies entirely outside Q, so that by (25)

$$\int_{\sigma_N} | F(\omega) + R_N(\omega)|^2 d\omega \leqq \frac{\gamma_N}{2\pi l_N} \int_{t \notin Q} |f(t)|^2 dt. \tag{26}$$

We expand the left-hand side of (26) and sum on N. We conclude that subjecting $f(t)$ to the requirements $(f, \psi_i) = 0,\ i = 1, \cdots, M$, where ψ_i are fixed functions in L^2, each of the form $\overline{h_N(k/l_N - t)}$ for some N and k, and $M \leqq \sum_N \{ [cl_N] + 2q \} \leqq [c \sum_N l_N] + 2pq = [c] + 2pq$, ensures

$$\begin{aligned} \sum_N \left\{ \int_{\sigma_N} | F(\omega)|^2 d\omega + \int_{\sigma_N} | R_N(\omega)|^2 d\omega + 2 \operatorname{Re} \int_{\sigma_N} F(\omega) \overline{R_N(\omega)}\, d\omega \right\} & \\ \leqq \sum_N \frac{\gamma_N}{2\pi l_N} \int_{t \notin Q} |f(t)|^2 dt. & \end{aligned} \tag{27}$$

Now $\sum_N \int_{\sigma_N} |F(\omega)|^2 d\omega = \|F\|^2 = \|f\|^2$. By Schwarz's inequality, (24), and the definition of ϵ,

$$
\begin{aligned}
&\left|\sum_N 2\,\mathrm{Re}\int_{\sigma_N} F(\omega)\overline{R_N(\omega)}\,d\omega\right| \\
&\qquad \leqq 2\sum_N \left\{\int_{\sigma_N} |F(\omega)|^2 d\omega\right\}^{1/2}\left\{\int_{\sigma_N} |R_N(\omega)|^2 d\omega\right\}^{1/2} \\
&\qquad \leqq 2\left\{\sum_N\int_{\sigma_N} |F(\omega)|^2 d\omega\right\}^{1/2}\left\{\sum_N\int_{\sigma_N} |R_N(\omega)|^2 d\omega\right\}^{1/2} \\
&\qquad \leqq 2\epsilon\|F\|^2\left\{2p^2 + \sum_N\, [1/l_N]\right\}^{1/2} = \|f\|^2/2,
\end{aligned}
\tag{28}
$$

so that from (27)

$$
\|f\|^2/2 \leqq \left\{\int_{t\notin Q} |f(t)|^2 dt\right\}\sum_N \frac{\gamma_N}{2\pi l_N} = \{\|f\|^2 - \|Df\|^2\}\sum_N \frac{\gamma_N}{2\pi l_N}\,. \tag{29}
$$

Setting $J = (\sum_N \gamma_N/l_N - \pi)/(\sum_N \gamma_N/l_N) < 1$ we may rewrite (29) as

$$
\|Df\|^2/\|f\|^2 \leqq J,
$$

whence by (19) $\lambda_{[c]+2pq} \leqq J < 1$. The constant J depends on P (normalized to have measure 2π) since the numbers γ_N and l_N do, but it does not depend on Q. Theorem 3 is established.

Theorem 4. *Under the hypotheses of Theorem* 3, $\lambda_{[c]-2pq} \geqq 1 - J > 0$.

Proof. We normalize P to have measure 2π, as in Theorem 3, and let σ_N, l_N, ϵ, $\chi_N(\omega)$, $h_N(t)$, γ_N, and J be as defined there. As in Theorem 2, we will base our proof on the modification of the Weyl-Courant lemma which asserts that

$$
\lambda_{[c]-2pq} \geqq \inf_{f\in \mathscr{B}(P);\, f\in S} \frac{\|Df\|^2}{\|f\|^2}, \tag{30}
$$

where S is a subspace of dimension at least $[c] - 2pq + 1$.

For each $N = 1, \cdots, p$ we let $I(N)$ be the set of integers k for which the interval $|t - k/l_N| < 1/2l_N$ is contained in Q, and we define S as the subspace generated by those members of $\mathscr{B}(P)$ whose Fourier transforms are the functions $\chi_N(\omega)(2\pi l_N)^{-1/2}\exp(-i\omega k/l_N)$ with $k \in I(N)$. By Lemma 1, applied with a change of scale, the number of integers in $I(N)$ exceeds $cl_N - 2q$, so that the total number of generators exceeds $\sum_{N=1}^{p}(cl_N - 2q) = c - 2pq$ and, being an integer, is no smaller than $[c] - 2pq + 1$. Since the generators form an orthonormal set, the dimension of S is at least $[c] - 2pq + 1$, as is required.

If $f \in S$, its Fourier transform, by definition, satisfies

$$F(\omega) = \sum_{k \in I(N)} a_{N,k} \exp(-i\omega k/l_N)(2\pi l_N)^{-1/2}, \qquad \omega \in \sigma_N,$$

and because the above exponentials are orthonormal over σ_N we find

$$\|F\|^2 = \sum_{N=1}^{p} \sum_{k \in I(N)} |a_{N,k}|^2. \tag{31}$$

Introducing the functions $g_N(x)$ of (20) and $R_N(\omega)$ of (22) we obtain, as in (21),

$$\begin{aligned} (l_N)^{-1/2} g_N(k/l_N) &= \int_{\sigma_N} F(\omega)(2\pi l_N)^{-1/2} \exp(i\omega k/l_N)\, d\omega \\ &\quad + \int_{\sigma_N} R_N(\omega)(2\pi l_N)^{-1/2} \exp(i\omega k/l_N)\, d\omega, \end{aligned} \tag{32}$$

and we denote by $c_{N,k}$ the last term in (32). Another appeal to the orthonormality of $(2\pi l_N)^{-1/2}\exp(i\omega k/l_N)$ on σ_N, combined with (24), shows, for $k \in I(N)$,

$$(l_N)^{-1/2} g_N(k/l_N) = a_{N,k} + c_{N,k}, \tag{33}$$

$$\sum_{k \in I(N)} |c_{N,k}|^2 \leqq \int_{\sigma_N} |R_N(\omega)|^2 d\omega \leqq \epsilon^2 \{[1/l_N] + 2p\} \|F\|^2. \tag{34}$$

Now from (33) and (25), together with the definition of $I(N)$,

$$\sum_{k \in I(N)} |a_{N,k} + c_{N,k}|^2 = l_N^{-1} \sum_{k \in I(N)} |g_N(k/l_N)|^2 \leqq \frac{\gamma_N}{2\pi l_N} \int_{t \in Q} |f(t)|^2 dt. \tag{35}$$

We expand the left-hand side of (35) and sum on N. Using Schwarz's inequality, (31), (32), Parseval's theorem, and the definition of ϵ, we find, just as in (28),

$$\left| \sum_{N=1}^{p} 2 \operatorname{Re} \sum_{k \in I(N)} a_{N,k} \overline{c_{N,k}} \right| \leqq \|f\|^2/2,$$

so that, by (31) and Parseval's theorem,

$$\|f\|^2/2 \leqq \|Df\|^2 \sum_N \frac{\gamma_N}{2\pi l_N}, \tag{36}$$

or

$$\|Df\|^2/\|f\|^2 \geqq 1 - J,$$

for all $f \in S$. By (30), $\lambda_{[c]-2pq} \geqq 1 - J > 0$. Theorem 4 is established.

LEMMA 2. *Let P be any fixed set of measure 2π, and Q_k be a sequence of* *sets, each the union of q intervals, and of total measure $c_k \to \infty$. Then the number of eigenvalues of $B_P D_{Q_k} B_P$ contained in any fixed subinterval of the unit interval cannot remain bounded as $k \to \infty$.*

Proof. To simplify notation we will write B and D_k instead of B_P and D_{Q_k}, respectively.

M. Rosenblum has shown [to appear] that the Wiener-Hopf operator $Af = (2\pi)^{-1/2} \int_0^\infty h(t-x) f(x)\, dx,\ t \geqq 0,$ on $L^2(0, \infty)$ may be taken into a Toeplitz transformation by a unitary mapping. If the Fourier transform $H(\omega)$ of $h(t)$ is bounded, the spectrum of A may then be determined by the results of [1]. In particular, when $H(\omega)$ is real, this method shows the spectrum of A to consist of all numbers between ess sup $H(\omega)$ and ess inf $H(\omega)$. I am indebted to H. Widom for suggesting this argument, on which our proof will be based.

Letting S be the set $t \geqq 0$, D be the projection D_S, and $H(\omega)$ be the characteristic function of P, the operator DBD coincides with A. Thus the spectrum of DBD consists of all $0 \leqq \lambda \leqq 1$.

We now show that DBD and BDB have the same spectra. For by definition, if $0 < \lambda < 1$ is not in the spectrum of BDB, the operator $BDB - \lambda B$ has an inverse bounded by some M. Then from the equation $DBDf - \lambda Df = g$, using the idempotency of projections and their boundedness, we obtain

$$|\lambda|\, \|Df\| - \|g\| \leqq \|\lambda Df + g\| = \|DBDf\| \leqq \|BDf\| = \|(BDB - \lambda B)^{-1} Bg\| \leqq M\|g\|.$$

But since $Df = (DBD - \lambda D)^{-1} g$, this shows $DBD - \lambda D$ to have an inverse bounded by $(M+1)/|\lambda|$, contradicting the fact that λ is in the spectrum of DBD. Since the spectrum of BDB is a closed subset of the unit interval, it consists of all $0 \leqq \lambda \leqq 1$.

Next let $\tau_i(k)$, $i = 1, \cdots, q$ be the (i)th interval of Q_k, counted from the left. Let r be the least integer for which the length of $\tau_r(k)$ becomes unbounded as $k \to \infty$; $1 \leqq r \leqq q$, since by assumption $c_k \to \infty$. Because the eigenvalues of BD_kB are not affected by a translation of Q_k, we may suppose that the left-hand endpoint of $\tau_r(k)$ coincides with the origin. By choosing a suitable subsequence of the sets Q_k, we may also suppose that each $\tau_i(k)$, $i < r$, converges to a limit (possibly to infinity). Let S' be the set of all points on $-\infty < t < 0$ which are limits of points in $\tau_i(k)$, $i < r$, and let D' denote the projection $D_{S'}$. Then for any fixed $\phi \in L^2$,

$$\|(D + D')\phi - D_k\phi\| \to 0 \tag{37}$$

as $k \to \infty$.

Since by definition of r the lengths of $\tau_i(k)$, $i < r$, are bounded, S' has finite measure, so the operator $BD'B$ is completely continuous. By a theorem of Weyl [5, p. 367], the addition of such an operator to the bounded self-adjoint BDB does not change any limit point of the spectrum. Thus the spectrum of $B(D+D')B$ contains all $0<\lambda<1$. Consequently given λ, $0<\lambda<1$, and $\epsilon>0$, there exists a function Bf_ϵ with $\|Bf_\epsilon\|=1$ and

$$\|B(D+D')Bf_\epsilon-\lambda Bf_\epsilon\|<\epsilon/2. \tag{38}$$

Now by (37) we may choose k_0 so that for all $k>k_0$, $\|(D+D')Bf_\epsilon-D_kBf_\epsilon\|<\epsilon/2$. Since $\|B\|\leqq 1$, this implies $\|B(D+D')Bf_\epsilon-BD_kBf_\epsilon\|<\epsilon/2$, and combining this with (38) yields $\|BD_kBf_\epsilon-\lambda Bf_\epsilon\|\leqq\epsilon$. Then by the triangle inequality

$$\|BD_kBf_\epsilon\|\geqq\lambda-\epsilon, \tag{39}$$

and since BD_kB is bounded by 1,

$$\|(BD_kB)^2f_\epsilon-\lambda BD_kBf_\epsilon\|\leqq\|BD_kBf_\epsilon-\lambda f_\epsilon\|\leqq\epsilon. \tag{40}$$

But by [5, p. 234] the eigenfunctions $\phi_i^{(k)}$ of BD_kB are sufficient to expand any element in its range. Thus we may write

$$BD_kBf_\epsilon=\sum_i a_i\phi_i^{(k)}, \tag{41}$$

whence by (39)

$$(\lambda-\epsilon)^2\leqq\|BD_kBf_\epsilon\|^2=\sum_i|a_i|^2, \tag{42}$$

and

$$(BD_kB)^2f_\epsilon=\sum_i a_iBD_kB\phi_i^{(k)}=\sum_i a_i\lambda_i^{(k)}\phi_i^{(k)}, \tag{43}$$

where $\lambda_i^{(k)}$ is the (i)th eigenvalue of BD_kB. Introducing (41) and (43) into (40) yields by (42)

$$\epsilon^2\geqq\sum_i|a_i|^2|\lambda_i^{(k)}-\lambda|^2\geqq\inf_i|\lambda_i^{(k)}-\lambda|^2(\lambda-\epsilon)^2.$$

Choosing ϵ sufficiently small, we conclude that every neighborhood of λ will contain an eigenvalue of BD_kB for all $k>k_0$.

To complete the proof of the lemma, given any subinterval I of the unit interval, and any integer N, we divide I into N disjoint subintervals. By the above, each of these subintervals will contain an eigenvalue of BD_kB for all k sufficiently large. Thus the number of eigenvalues of BD_kB contained in I cannot remain bounded as $k\to\infty$. Lemma 2 is established.

COROLLARY 2. *Under the hypotheses of Theorem* 3, *with any fixed integer* N,

$$\lambda_{[c]-N}\leqq J_1<1,$$

where J_1 *is a constant depending on* P (*suitably normalized*), q, *and* N, *but not on* Q.

Proof. We will argue by contradiction. Accordingly, let us suppose that for a given P, normalized as in Theorem 3 to have measure 2π, there exists a sequence of sets S_k, each the union of q intervals and of total measure c_k, for which $\lambda_{[c_k]-N} \to 1$. To simplify notation, let us denote the projections B_P and D_{S_k} by B and D_k respectively, and the eigenvalue $\lambda_{[c_k]-N}$ of BD_kB by λ_k^*.

By Theorem 3, $\lambda_{[c_k]+2pq} \leqq J < 1$, and we are assuming $\lambda_k^* \to 1$. Thus for all K sufficiently large, the interval $J \leqq x \leqq (J+1)/2$ will contain no more than $2pq + N$ eigenvalues of BD_kB. We conclude by Lemma 2 that the measures c_k of S_k must be bounded: $c_k < C$.

Now let $\psi^{(k)}(t)$ be the eigenfunction of BD_kB corresponding to λ_k^*, normalized so that $\|\psi^{(k)}\| = 1$. We find, as in the transformations leading to (2)

$$\int_{S_k} |\psi^{(k)}(t)|^2 dt = \lambda_k^* \to 1. \tag{44}$$

Consequently, for one of the q subintervals of S_k, which we denote by $\tau(k)$,

$$\int_{\tau(k)} |\psi^{(k)}(t)|^2 dt > 1/2q. \tag{45}$$

Since the eigenvalues of BD_kB are not affected by a translation of S_k along the t-axis, we may suppose that the left-hand endpoint of $\tau(k)$ coincides with the origin. As we showed in the preliminary remarks, it follows from the normalization $\|\psi^{(k)}\| = 1$ of $\psi^{(k)} \in \mathscr{B}(P)$ that the functions $\psi^{(k)}(t)$ form a uniformly bounded family of analytic functions in any horizontal strip including the real axis, thus a normal family there. We may therefore suppose that

$$\psi^{(k)}(t) \to \psi(t), \tag{46}$$

uniformly on compact subsets of the t-axis, for this may always be ensured by choosing a suitable subsequence; $\psi(t)$ is then also analytic. For the same reason we may suppose that each subinterval of S_k converges to a limit (possibly to infinity). Since $c_k < C$, this limit of S_k has finite measure, so its complement includes some finite interval I; thus there exists k_0 such that I is disjoint from every S_k, $k > k_0$. Then for $k > k_0$, by the normalization of $\psi^{(k)}$ and (44),

$$\int_I |\psi^{(k)}(t)|^2 dt \leqq \int_{t \notin S_k} |\psi^{(k)}(t)|^2 dt = 1 - \int_{S_k} |\psi^{(k)}(t)|^2 dt$$
$$= 1 - \lambda_k^* \to 0,$$

whence by (46)

$$\int_I |\psi(t)|^2 dt = 0,$$

or $\psi(t) = 0$, $t \in I$. The analyticity of $\psi(t)$ then implies $\psi(t) \equiv 0$. On the other hand, let $0 < t < \alpha$ be the limit interval of $\tau(k)$. Since the functions $\psi^{(k)}(t)$ are uniformly bounded, (45) implies $\alpha \neq 0$, and $c_k < C$ implies $\alpha < C$. We may then again apply (46) on the finite interval $0 \leqq t \leqq \alpha$ to conclude $\int_0^\alpha |\psi(t)|^2 > 1/2q$, whence $\psi(t) \not\equiv 0$, and we have reached a contradiction. Corollary 2 is established.

Corollary 3. *Under the hypotheses of Theorem* 3, *with any fixed integer* N *and* $c \geqq 1$,

$$\lambda_{[c]+N} \geqq J_2 > 0,$$

where J_2 *is a constant depending on* P (*suitably normalized*), q, *and* N, *but not on* Q.

Proof. The restriction $c \geqq 1$ is necessary, since as $c \to 0$ every eigenvalue approaches 0. As in the proof of Corollary 2, we will argue by contradiction. Accordingly, let us suppose that for a given P, normalized as in Theorem 3 to have measure 2π, there exists a sequence of sets S_k, each the union of q intervals and of total measure c_k, for which $\lambda_{[c_k]+N} \to 0$. To simplify notation, we denote the projections B_P and D_{S_k} by B and D_k respectively, and the eigenvalue $\lambda_{[c_k]+N}$ of BD_kB by λ_k'.

By Theorem 4, $\lambda_{[c_k]-2pq} \geqq J' > 0$, and we are assuming $\lambda_k' \to 0$. Thus for all k sufficiently large, the interval $J'/2 \leqq x \leqq J'$ will contain no more than $2pq + N$ eigenvalues of BD_kB. By Lemma 2, the measures c_k of S_k must be bounded: $c_k < C$. Then $[c_k] + N < [C] + N$ so that

$$\lambda_k' \geqq \lambda_{[C]+N}^{(k)}, \tag{47}$$

where $\lambda_{[C]+N}^{(k)}$ is the eigenvalue $\lambda_{[C]+N}$ of BD_kB. Since S_k has total measure $c_k \geqq 1$, at least one of its q subintervals will have measure exceeding $1/2q$. Because the eigenvalues of BD_kB are not affected by a translation of S_k, we may suppose S_k to include the interval I: $0 \leqq t \leqq 1/2q$. Letting T_k denote the remainder of S_k, we may write $BD_kB = BD_IB + BD_{T_k}B$. As we showed in the preliminary remarks, the latter operator is positive, thus by an application of the Weyl-Courant lemma [5, p. 239],

$$\lambda_{[C]+N}^{(k)} \geqq \lambda_{[C]+N}^{(I)},$$

where $\lambda_{[C]+N}^{(I)}$ is the eigenvalue $\lambda_{[C]+N}$ of BD_IB, and hence is positive. Combining the above with (47) yields $\lambda_k' \geqq \lambda_{[C]+N}^{(I)} > 0$, and since the right-hand quantity is independent of k, this contradicts our assumption that $\lambda_k' \to 0$. Corollary 3 is established.

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